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MAT 500

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**Project 1: Proof Conceptions of College Calculus Students**

**Introduction**

There has been “increasing awareness that reasoning is central to mathematics and mathematics learning” (Yackel & Hanna, 2003, p. 227) among education researchers, which raises questions about student conceptions of reasoning, argumentation and mathematical proof.

Mathematicians and mathematics education researchers have consistently asserted the crucial roles deductive reasoning and proof play in discovering, communicating, verifying, understanding, and systematizing mathematics (Hanna, 2000; Ko, 2008; Thurston, 1998). In response to the many affirmations of the importance of proof for learning and understanding mathematics, there has been extensive study of students’ conceptions of mathematical proof, their abilities to construct and understand proofs, and the frequency with which inductive evidence is accepted as sufficient verification of mathematical conjectures (Bell, 1976; Healy & Hoyles, 2000; Ko, 2008; Stylianides, 2009; Varghese, 2009).

These studies have provided valuable insights into “what types of reasoning students are capable of at various age and grade levels, how their notions of reasoning and proof develop over time, and what limitations in reasoning they exhibit” (Yackel & Hanna, 2003, p. 230). Researchers have focused on different populations, ranging from elementary school students to advanced university mathematics students, and they have yielded several consistent and useful findings. Broadly speaking, many students at all grade levels have difficulty with the processes of creating and evaluating deductive mathematical proofs, and many believe simple empirical arguments are convincing proofs.

The empiricist approach of testing many cases may be effective, although perhaps cumbersome, in some discrete systems in which only a finite number of possibilities exist. However, for mathematical conjectures which apply to infinite domains, naïve empiricism is only a way to provide fragmented evidence in support of a conjecture rather than a method for producing incontrovertible proof. Further, this approach is inconsistent with the views of mathematicians and the demands of the CCSSM, and in situations where a counterexample is not immediately obvious, the naïve empiricist strategy can quickly lead students to make false conclusions.

Past research has focused primarily on three groups: students in high school geometry courses, pre-service and in-service teachers, and students in advanced undergraduate courses who have received formal instruction in proof writing. However, little attention has been given to examining students’ understanding of proof after the completion of a high school geometry course, but before taking a course explicitly focused and dependent upon proof. The purpose of this study is to fill this gap in the literature by examining university students enrolled in Calculus courses.

The following question is used to frame the study:

*What forms of empirical arguments accepted by college calculus students as proofs of mathematical conjectures?*

The Common Core State Standards for Mathematics, or CCSSM, which have now been adopted by 45 states and three U.S. territories, emphasizes the importance of reasoning and proof in K-12 mathematics education. The CCSSM states that secondary school students must learn to “construct viable arguments and critique the reasoning of others,” to “reason abstractly and quantitatively,” and to begin “using more precise definitions and developing careful proofs” (National Governors Association, 2010, p. 74). The NCTM Standards (2000) make the more forceful claims that “systematic reasoning is a defining feature of mathematics” (p. 57), and that secondary students should “recognize reasoning and proof as fundamental aspects of mathematics” and “develop and evaluate mathematical arguments and proofs” (p. 342).

Studying students in an introductory college calculus courses can contribute to evaluating whether recent high school graduates have an understanding of mathematical reasoning and proof consistent with what is described in the CCSSM and NCTM standards for mathematics. Through examining how students perceive mathematical arguments, researchers and practitioners can more effectively develop and evaluate targeted interventions for K-12 mathematics students.

**Data Collection**

Fifty-five participants were solicited to complete a written questionnaire from an introductory differential calculus course at mid-sized northeastern university. This population fills a gap in the literature by examining students’ conceptions of reasoning and proof after the completion of a high school geometry course, but before enrollment in a course focused on the creation and evaluation of mathematical proofs.

At this level in a typical undergraduate mathematics sequence, students are assumed to not have received substantial formal instruction in the creation and evaluation of mathematical proofs beyond what they may have received in secondary school. Further, for many students, Calculus may be their last purely mathematical formal educational experience, and thus their conceptions of mathematical proof may reach their peak development at this level.

All data were collected and analyzed within a cognitivist theoretical framework, and it is therefore assumed that conceptual understanding can be reasonably measured through voluntary responses to written and verbal prompts. Many previous investigations related to student understanding of mathematical proof have similarly made this assumption, and therefore it is believed to be a suitable framework for the present study.

An eight-item written survey, titled the Mathematical Reasoning Questionnaire, was used as the primary data collection instrument. The questionnaire is primarily comprised of items used during previous studies of the proof conceptions of secondary school and undergraduate students. In the present study, only results from the second survey question, created by Martin and Harel (1989) for a study of the proof conceptions of pre-service elementary teachers, will be analyzed. The question, shown in Figure 1, is a Likert-style item that was created to test the theory that for many students, the perceived validity of an empirical argument is dependent upon relatively superficial characteristics. Participants were asked to rate four arguments for elementary integer divisibility theorem. The arguments to be rated are structured to follow the proof schemes outlined in Table 1.

|  |  |
| --- | --- |
| **Proof Scheme** | **Description** |
| Argument A: Single example | A single example supporting relationship is used as proof. |
| Argument B: Extreme example | An example of the relationship holding for case perceived as extreme, such as a large number, is used as proof. |
| Argument C: Example and non-example | The combination of evidence from a supporting example and a non-example are used as proof. |
| Argument D: Pattern | A list of many examples supporting the relationship is used as proof. |

*Table 1: Survey Question Two Argument Classifications*

**(2)** Consider the following statement and arguments. Using the space provided, rate how well each argument convinces you of the truth of the statement on a scale of 1 to 4 using the following criteria:

1: Not a convincing mathematical proof

2: Slightly convincing mathematical proof

3: Mostly convincing mathematical proof

4: Completely convincing mathematical proof

***Statement***: If the sum of the digits of any whole number is divisible by 3, then the number itself is divisible by 3.

***Arguments:***

**A.** \_\_\_\_\_\_\_ The sum of the digits of 123 is 6, which is divisible by 3. The number 123 is also divisible by 3.

**B.** \_\_\_\_\_\_\_ We can pick any number so the sum of its digits is divisible by 3, say 731234082. The sum of the digits is 30, which is divisible by 3, and the number itself is also divisible by 3.

**C.**\_\_\_\_\_\_\_ 31 is not divisible by 3, and the sum of its digits is 4, which is not divisible by 3. On the other hand, 36 is divisible by 3, and the sum of its digits is 9, which is divisible by 3.

**D.** \_\_\_\_\_\_\_ If we list several numbers that we know are divisible by 3, say 3, 6, 12, 15, 18, 24, 36, 48, 1002, 1008, and so on, we can see that the sums of their digits are always divisible by 3.

*Figure 1: Survey Question Two*

**Analysis**

Consistent with the methods of Martin and Harel (1989), ratings of “1” or “2” were categorized as not supporting the argument and ratings of “3” or “4” were categorized as supporting the argument. This allows each response to be viewed as a Bernoulli trial in which supporting the argument is considered a “success.” Using data collected by Martin and Harel (1989), shown in Table 2, beta prior distributions for the proportion of students supporting each of the arguments, , were created. These priors will be used as the basis for a Bayesian analysis to determine if the students sampled for this study are biased toward accepting empirical arguments as mathematical proofs.

|  |  |  |  |
| --- | --- | --- | --- |
| **Argument** | **Number Supporting** | **Number Not Supporting** | **Prior Distribution** |
| **A** | 65 | 36 | Beta(65, 36) |
| **B** | 56 | 45 | Beta(56, 45) |
| **C** | 69 | 32 | Beta(69, 32) |
| **D** | 75 | 26 | Beta(75, 26) |

*Table 2: Data from Martin and Harel and Prior Distributions*

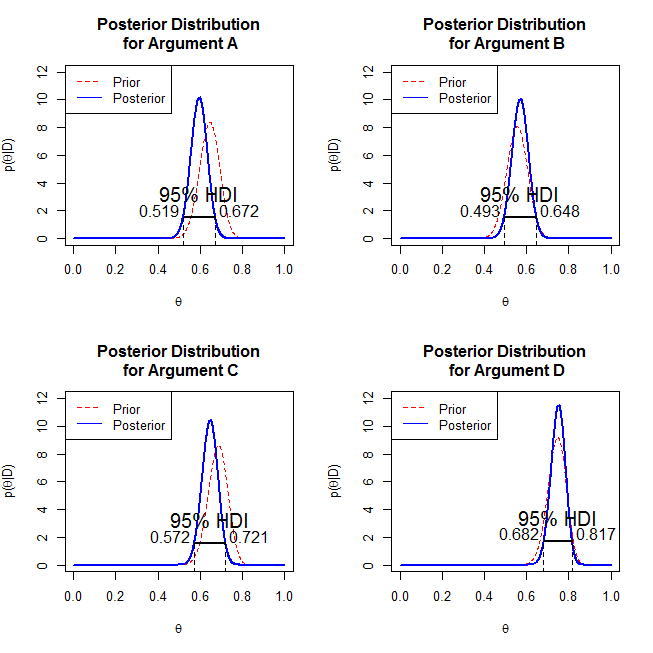
The prior distributions for are tightly distributed about the proportions supporting each argument reported by Martin and Harel (1989), which expresses a high degree of confidence in their findings. Given that subsequent researchers, such as Chazan (1993) and Harel and Sowder (1998), found similar biases for empirical proof schemes, this confidence is believed to be warranted.

Using the new sample of 55 students and the prior distributions from Table 2, the posterior distributions for given the data, can be found analytically. Given a prior distribution Beta(a, b) and a sample of N students with Z supporting the argument, the posterior distribution for the proportion supporting each argument is . The sample data and posterior distributions for each argument are shown in Table 3.

|  |  |  |  |
| --- | --- | --- | --- |
| **Argument** | **Number Supporting** | **Number Not Supporting** | **Posterior** |
| **A** | 28 | 27 | Beta(93, 63) |
| **B** | 33 | 22 | Beta(89, 67) |
| **C** | 32 | 23 | Beta(101, 55) |
| **D** | 42 | 13 | Beta(117, 39) |

*Table 3: Survey Item Two Data and Posterior Distributions*

Figure 2 shows the posterior densities for each of the four arguments. The prior distributions are shown in red for reference. For Arguments A, C, and D, the 95% highest density intervals, or HDIs, suggest that students are bias toward accepting the arguments because the intervals lie entirely above While it is credible to believe that students are not biased in support or against Argument B since lies in its HDI, there is evidence in favor of a supporting bias. , so the posterior suggests that it is considerably more likely that students are biased toward supporting Argument B.



*Figure 2: Posterior Distributions by Argument*

**Conclusions**

The data indicate that most introductory calculus students will accept some form of empirical evidence as a mathematical proof. Students were most strongly biased in favor of the pattern proof scheme, exhibited by Argument D. The posterior for this argument suggests that more than two-thirds of introductory calculus students accept a listing of several cases as a proof. Chazan (1993) and Martin and Harel (1989) similarly found that this was the form of empirical argument most commonly accepted by high school geometry students and pre-service teachers, respectively. Students also exhibit biases toward inappropriately classifying the testing of single cases, evidenced by Arguments A and B, and the testing of an example and counterexample, shown by Argument C, as acceptable forms of proofs. Martin and Harel (1989) and other researchers have similarly reported biases toward these forms of arguments.

These findings suggest that many students are lack a robust view of the role of empirical evidence in mathematics. However, they make no indication of how to address this issue, and there is currently little consensus about how the topic of proof should be incorporated into K-12 education or introductory calculus. Further research is needed to determine which opportunities to learn about proof currently exist in typical secondary school and introductory college curricula and how these opportunities shape student beliefs and problem solving approaches.

Additional research is also needed to determine which instructional methods can most effectively help students form an understanding of deductive proof and reasoning that is consistent with the recommendations of the NCTM and the Common Core State Standards. Maher and Martino (1996) have demonstrated that even elementary and middle school students can learn to approach proof as a necessarily deductive process when given structured guidance over a number of years, but such approaches have not been widely implemented due to time constraints and teacher inexperience with proof.

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